Introduction to Magnetohydrodynamics

part B of "Selected Topics in High Energy Astrophysics"

Łukasz Stawarz

Astronomical Observatory of the Jagiellonian University, Kraków (Poland)

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Part I

HYDRODYNAMICS

Preliminaries

Main assumption

We approximate the cosmic plasma as a continuous medium, treating it as an electrically neutral fluid. The first assumption, despite the typically collisionless nature of the plasma, necessitates the presence of electromagnetic fields to confine particles within the system (particles interact through electromagnetic fields in the form of plasma waves, rather than through binary collisions). The second assumption implies averaging over spatial and temporal scales larger than those set by the plasma and cyclotron frequencies of all plasma species (note that the inverse of the plasma frequency sets a characteristic time scale for collective oscillations, and is comparable to the time it takes a thermal particle to traverse a Debye length.)

General references for MHD tutorials and textbooks

- Kulsrud, R. M. (2005). ***Plasma Physics for Astrophysics*.**
- Ogilvie, G. I. (2016).**Lecture notes: Astrophysical fluid dynamics.** *arXiv e-prints*, Article arXiv:1604.03835, arXiv:1604.03835. https://doi.org/10.48550/arXiv.1604.03835
- Spruit, H. C. (2013).**Essential Magnetohydrodynamics for Astrophysics.** *arXiv e-prints*, Article arXiv:1301.5572, arXiv:1301.5572. https://doi.org/10.48550/arXiv.1301.5572

Let us define macroscopic scalar parameters of the jet fluid:

- **number density** n,
- **pressure** p,
- **internal energy density** (including the rest-mass energy density) ϵ,
- **enthalpy** $w = \epsilon + p$.

All of these quantities are measured in the rest frame of the fluid, i.e. per **proper** unite volume, and therefore should be called "the proper number density", "the proper pressure", etc. ("primitive variables"). Note that the **proper specific internal energy** ε is defined as

$$\epsilon = mnc^2 + \varepsilon \quad \text{so that} \quad w = mnc^2 + \varepsilon + p \tag{1}$$

where **the proper (rest) mass density** is mn for a mass of a fluid particle m.

The four-velocity of the fluid is $u^\mu = (\Gamma, \Gamma\,\beta^k)$, where $\vec{v} \equiv (\beta^k c)$ is the bulk 3-velocity, $\Gamma \equiv (1-\beta^2)^{-1/2}$ is the bulk Lorentz factor, and the indices $\mu = 0, 1, 2, 3$ and $k = 1, 2, 3$. In the fluid rest frame one has $u'^\mu = (1, 0, 0, 0)$.

Preliminaries

Stress-energy tensor

For an **ideal fluid** (no energy dissipation, etc.), and in the absence of external forces, the **fluid stress-energy tensor** is diagonal in the fluid rest frame, namely

$$\mathcal{T}^{\mu\nu} = w\, u^{\mu} u^{\nu} - p\, g^{\mu\nu} \quad , \tag{2}$$

where $g^{\mu\nu}$ is the metric tensor of the **Minkowski spacetime**, with the $(+---)$ signature adopted here.

The **particle flux four-vector** is simply

$$\mathcal{D}^{\mu} = n\, u^{\mu} \quad . \tag{3}$$

The following individual components of the stress-energy tensor and particle flux vector can be identified with, respectively,

- **the total energy density** $\mathcal{T}^{00} = w\,\Gamma^2 - p = \left(\epsilon + p\beta^2\right)\,\Gamma^2$
- **the energy flux density** $\mathcal{T}^{0k} = w\,\Gamma^2\,\beta^k$
- **the momentum flux density** $\mathcal{T}^{ik} = w\,\Gamma^2\,\beta^i\beta^k + p\,\delta^{ik}$
- **the particle number density** $\mathcal{D}^0 = n\,u^0 = n\,\Gamma$
- **the particle flux density** $\mathcal{D}^k = n\,u^k = n\,\Gamma\,\beta^k$

Ideal fluid

Conservation laws

The local conservation laws in relativistic ideal hydrodynamics are obtained from vanishing divergence of the stress-energy tensor of the fluid,

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 0 \tag{4}$$

(energy-momentum conservation), and of the particle flux,

$$\nabla_\mu \mathcal{D}^{\mu} = 0 \tag{5}$$

(particle conservation), where $\nabla_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\partial_t, \nabla_k)$ is the covariant differential operator.

Ideal fluid

In Cartesian coordinates

The conservation laws $\nabla_\mu \mathcal{D}^\mu$ and $\nabla_\mu \mathcal{T}^{\mu\nu} = 0$, in Cartesian coordinates, become

$$\partial_t(n\Gamma) + \partial_i(n\Gamma\beta^i c) = 0 \tag{6}$$

$$\partial_t(w\Gamma^2 - p) + \partial_i(w\Gamma^2\beta^i c) = 0 \tag{7}$$

$$\partial_t(w\Gamma^2\beta^k) + \partial_i(w\Gamma^2\beta^i\beta^k c + pc\delta^{ik}) = 0 \tag{8}$$

Let us now define the following quantities **measured in the laboratory frame**:

- **the rest mass density** $\rho \equiv m\mathcal{D}^0 = mn\Gamma$
- **the total energy density** $U \equiv \mathcal{T}^{00} = w\Gamma^2 - p$
- **the momentum density vector** $\vec{P} \equiv \mathcal{T}^{0k}/c = w\Gamma^2\vec{\beta}/c$

(note that $\rho \neq mn$). With such, the conservations laws become

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{9}$$

$$\partial_t U + \partial_i(Uv^i + pv^i) = 0 \tag{10}$$

$$\partial_t P^k + \partial_i(P^k v^i + p\delta^{ik}) = 0 \tag{11}$$

Ideal fluid

Newtonian quantities

In the non-relativistic limit, it is convinient to subtract the rest-mass density from the total energy density, $\tilde{U} \equiv U - \rho c^2$, noting that this will not affect the energy conservation law, namely $\partial_t \tilde{U} + \partial_i(\tilde{U} v^i + p v^i) = 0$. With such, **Newtonian counterparts** for the quantities ρ, $\tilde{U}$, and $\vec{P}$, can be found by series expansions for non-relativistic bulk velocity $\beta \to 0$ (and so $\Gamma \simeq 1 + \frac{1}{2}\beta^2$ and $\Gamma^2 \simeq 1 + \beta^2$), assuming moreover "cold plasma" $mnc^2 \gg \varepsilon + p$ in the momentum equation, namely

$$\rho = mn\Gamma \;\to\; mn(1 + \frac{1}{2}\beta^2) + \mathcal{O}(\beta^4) \;\simeq mn \tag{12}$$

$$\begin{aligned} \tilde{U} &= \rho c^2(\Gamma - 1) + \varepsilon\Gamma^2 + p(\Gamma^2 - 1) \;\to\; \frac{1}{2}\rho c^2\beta^2 + \varepsilon(1 + \beta^2) + p\beta^2 + \mathcal{O}(\beta^4) \\ &\simeq \frac{1}{2}\rho v^2 + \varepsilon \end{aligned} \tag{13}$$

$$\begin{aligned} \vec{P} &= (mnc^2 + \varepsilon + p)\Gamma^2\vec{\beta}/c \simeq mnc^2\Gamma^2\vec{\beta}/c = \rho c^2\Gamma\vec{\beta}/c \;\to\; \rho c^2\vec{\beta}/c + \mathcal{O}(\beta^3) \\ &\simeq \rho\vec{v} \qquad \textit{(in the momentum equation only!)} \end{aligned} \tag{14}$$

All in all, the set of equations describing **non-relativistic ideal hydrodynamics**, can therefore be written in the form of the **Euler conservation laws**:

$$\textbf{mass} \qquad \partial_t \rho + \partial_i(\rho v^i) = 0 \tag{15}$$

$$\textbf{energy} \qquad \partial_t(\frac{1}{2}\rho v^2 + \varepsilon) + \partial_i(\frac{1}{2}\rho v^2 + \varepsilon + p)v^i = 0 \tag{16}$$

$$\textbf{momentum} \qquad \partial_t(\rho v^k) + \partial_i(\rho v^k v^i + p\delta^{ik}) = 0 \tag{17}$$

Note the general form of these equation "$\partial_t\,\mathrm{stuff} + \vec{\nabla}\cdot\mathrm{flux\ of\ stuff} = 0$", which may be therefore expressed in the integral forms, by integrating over volume $\mathcal{V}$ and using the **Gauss theorem**

$$\int \left(\vec{\nabla}\cdot\vec{F}\right)\, d\mathcal{V} = \int_{\partial\mathcal{V}} \vec{F}\cdot d\vec{\mathcal{S}} \tag{18}$$

where the volume's surface $\partial\mathcal{V} \equiv \mathcal{S}$ with the outward-pointing normal unit vector $\hat{n}$ for each differential surface, $d\vec{\mathcal{S}} = \hat{n}\, d\mathcal{S}$.

Ideal fluid

Convective derivative

Now, let us re-write the mass conservation law, $\partial_t \rho + \partial_i(\rho v^i) = 0$, as

$$(\partial_t + v^i \partial_i)\rho = -\rho\, \partial_i v^i \tag{19}$$

and denote the **convective derivative** (‘comoving derivative’, ‘material derivative’, ‘substantial derivative’,...) as

$$D_t \equiv \partial_t + v^i \partial_i \tag{20}$$

which is measuring the changes of a quantity as it follows a fluid flow:

$$\begin{aligned}
\Delta f &= f(t + \Delta t, \vec{x} + \vec{v}\Delta t) - f(t, \vec{x}) \\
&\simeq \Big[f(t, \vec{x}) + \Delta t\ \partial_t f(t, \vec{x}) + \Delta t\, \vec{v} \cdot \vec{\nabla} f(t, \vec{x}) \Big] - f(t, \vec{x}) \\
&= \Delta t \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) f(t, \vec{x}) && (21) \\
\rightarrow \quad d_t f &\equiv \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) f(t, \vec{x}) && (22)
\end{aligned}$$

Ideal fluid

Lagrangian form

Accordingly, all the combining and re-arranged Euler conservation laws may be written in the compact **Lagrangian form**

$$
\begin{aligned}
&\textbf{mass} && D_t\,\rho = -\rho\,\vec{\nabla}\cdot\vec{v} && (23)\\
&\textbf{energy} && D_t\left(\frac{\varepsilon}{\rho}\right) = -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} && (24)\\
&\textbf{momentum} && D_t\,\vec{v} = -\frac{1}{\rho}\,\vec{\nabla}p && (25)
\end{aligned}
$$

Continuity equation

Various formulations

Let's look at various form form of the law for the conservation of fluid mass, aka the continuity equation:

$$\textbf{Euler} \qquad \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{26}$$

$$\textbf{Gauss} \qquad \partial_t \int d\mathcal{V}\, \rho = \int_{\partial \mathcal{V}} d\vec{\mathcal{S}} \cdot (\rho \vec{v}) \tag{27}$$

$$\textbf{Lagrange} \qquad D_t \rho = -\rho \vec{\nabla} \cdot \vec{v} \tag{28}$$

- the first one says that the time derivative of a mass density at a given place is balanced by the divergence of the mass density flux
- the second one is telling us that the variation of mass in the volume $\mathcal{V}$ must be entirely due to the -in or -outflow of mass through the volume's surface $\partial\mathcal{V} \equiv \mathcal{S}$
- the third one is telling us that changes in the fluid density *along* the flow, are due to the fluid compression $-\vec{\nabla} \cdot \vec{v}$.

Continuity equation

Compressible/Incompressible fluid

Let's re-write the continuity equation as

$$\frac{\partial_t \rho}{\rho} + \vec{v} \cdot \frac{\vec{\nabla} \rho}{\rho} = -\vec{\nabla} \cdot \vec{v} \tag{29}$$

where the term $\vec{v} \cdot \vec{\nabla} \rho$ describes **advection of the fluid element with velocity** $\vec{v}$, and $-\vec{\nabla} \cdot \vec{v}$ is the **compression term**:

- $\vec{\nabla} \cdot \vec{v} < 0$ corresponds to the case of converging flows (fluid compression);
- $\vec{\nabla} \cdot \vec{v} < 0$ corresponds to the case of diverging flows (fluid dilation);
- $\vec{\nabla} \cdot \vec{v} = 0$ is the condition for incompressible fluid; for such, the conservation equation implies in particular $D_t \, \rho = 0$, meaning that the fluid density is conserved along the flow.

Let us recall the **fundamental thermodynamic relation** (i.e., mathematical summation of the first and second law of thermodynamics):

$$dU = T\,dS - p\,d\mathcal{V} \tag{30}$$

where T is the **temperature**, $U = \varepsilon\mathcal{V}$ is the internal energy of a fluid, and S is the **fluid entropy**. Noting that $\rho = M/\mathcal{V}$, where M is the total mass within the volume $\mathcal{V}$, one therefore has $U = \varepsilon M/\rho$, and hence, assuming M is constant,

$$d\left(\frac{\varepsilon}{\rho}\right) = T\,ds - p\,d\left(\frac{1}{\rho}\right) \tag{31}$$

where $s \equiv S/M$ is the **specific entropy**, i.e. fluid entropy per unit mass.

Energy equation

Entropy conservation

Now, let us note that $d(1/\rho) = -\rho^{-2} d\rho$, replace the total derivative with the convective derivative, $d \to D_t$, and recall the continuity equation in the Lagrangian form $D_t \rho = -\rho\, \vec{\nabla} \cdot \vec{v}$; with all of such we arrive at

$$D_t \left(\frac{\varepsilon}{\rho} \right) = T\, D_t s - \frac{p}{\rho}\, \vec{\nabla} \cdot \vec{v} \tag{32}$$

This, when compared with the energy equation in the Lagrangian form, implies that

$$D_t\, s = 0 \tag{33}$$

i.e., that the **specific entropy is conserved along the flow** (that is, following a volume element along the flow) or, in other words, that the ideal fluid is adiabatic.

Energy equation

Equation of state

Let us introduce the **equation of state** as the relation

$$p = (\hat{\gamma} - 1)\varepsilon \tag{34}$$

where $\hat{\gamma}$ is the adiabatic index, and recall again the energy equation in the Lagrangian form $D_t\left(\frac{\varepsilon}{\rho}\right) = -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v}$. From there it follows that

$$D_t\left(\frac{p}{\rho}\right) = -(\hat{\gamma} - 1)\,\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} = (\hat{\gamma} - 1)\,\frac{p}{\rho^2}\,D_t\rho \tag{35}$$

(assuming that $\hat{\gamma}$ is constant along the flow!), where we used again the continuity equation. This is equivalent to

$$D_t \ln p = \hat{\gamma}\,D_t \ln \rho \tag{36}$$

meaning that

$$D_t\left(\frac{p}{\rho^{\hat{\gamma}}}\right) = 0 \tag{37}$$

Energy equation

Various formulation

We have therefore equivalent equations

$$\begin{aligned}
\textbf{energy conservation} \quad D_t\left(\frac{\varepsilon}{\rho}\right) &= -\frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v} && (38)\\
\textbf{+ thermodynamic relation} \quad D_t\, s &= 0 && (39)\\
\textbf{+ equation of state} \quad D_t\left(\frac{p}{\rho^{\hat{\gamma}}}\right) &= 0 && (40)
\end{aligned}$$

Energy equation
Polytropic fluid

Note that the fundamental thermodynamic relation with the specific entropy conserved, $ds = 0$, reads as

$$d\left(\frac{\varepsilon}{\rho}\right) = -p\, d\left(\frac{1}{\rho}\right) \tag{41}$$

which, for the equation of state $p = (\hat{\gamma} - 1)\varepsilon$, may be re-arranged as

$$d \ln p = \hat{\gamma}\, d \ln \rho \tag{42}$$

meaning

$$p = K\, \rho^{\hat{\gamma}} \tag{43}$$

where K is the integration constant (*for an adiabatic process*). The above is called the **polytropic equation of state**. What is therefore a meaning of the statement $D_t(p/\rho^{\hat{\gamma}}) \equiv D_t K = 0$? And how does it relate to the specific entropy conservation $D_t s = 0$?

Energy equation

Forms of entropy

Recall that $\varepsilon = p/(\hat{\gamma} - 1) = K\rho^{\hat{\gamma}}/(\hat{\gamma} - 1)$, so that, *assuming K is a variable*,

$$d \ln \varepsilon = \hat{\gamma} \; d \ln \rho + d \ln K \tag{44}$$

Moreover, since $p = (\rho/m)kT$, the fluid temperature is

$$T = \frac{m}{k}\frac{p}{\rho} \tag{45}$$

Using these, the fundamental thermodynamic relation $d(\varepsilon/\rho) = T\,ds - p\,d(1/\rho)$ can be re-written as

$$\frac{m}{k}\; ds = \frac{1}{\hat{\gamma} - 1}\; d \ln K \tag{46}$$

meaning that K is, in fact, **a form of entropy**,

$$s = s_0 + \frac{k}{m\,(\hat{\gamma} - 1)}\; \ln K \tag{47}$$

and hence the conditions $D_t\, s = 0$ and $D_t\,(p/\rho^{\hat{\gamma}}) = 0$ are indeed equivalent.

Equation of motion

Inviscid fluid

Finally, let us look again at the momentum conservation equation in the Lagrangian form,

$$\rho\, D_t\, \vec{v} = -\vec{\nabla} p \tag{48}$$

which is clearly the **equation of motion** (recall the Newton's $m\, d_t \vec{v} = \vec{F}$), or the **Navier-Stokes equation** for an **ideal inviscid fluid**. It says that the element of a fluid will experience acceleration along the flow due to a force being the pressure gradient, $\vec{F} = -\vec{\nabla} p$. Note that any other force, such as gravity, or a Lorenz force, can therefore be incorporated as an additional term on the right-hand side of this equation.

Part II

MAGNETOHYDRODYNAMICS

Let us define the **electromagnetic potential**

$$\mathcal{A}^{\mu} = (\varphi, \vec{A}) \tag{49}$$

such that the magnetic field intensity $\vec{B} = \vec{\nabla} \times \vec{A}$ and the electric field $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t \vec{A}$. The **electromagnetic field tensor** is

$$\mathcal{F}_{\mu\nu} = \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu} \tag{50}$$

so that $\mathcal{F}^{0k} = -E^k$ and $\mathcal{F}^{ik} = -\varepsilon_{ikm}B^m$. Note that $\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu}$, and also the invariants $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = 2\,(B^2 - E^2) = inv$, along with $\varepsilon^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} = -8\,(\vec{E}\cdot\vec{B}) = inv$.

We moreover introduce the relativistic **four-vector electromagnetic current** as

$$\mathcal{J}^{\mu} = (c\,Q, \vec{j}) \tag{51}$$

where Q is the electric charge density, and $\vec{j}$ is the electric current density vector.

Maxwell's equations can now be formulated as

$$\nabla_\nu \mathcal{F}^{\mu\nu} = -\frac{4\pi}{c}\mathcal{J}^\mu \quad (52)$$

$$\epsilon^{\alpha\beta\mu\nu}\nabla_\beta \mathcal{F}_{\mu\nu} = 0 \quad (53)$$

In the Cartesian coordinates they obtain the familiar forms:

$$\textbf{Amper} \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \quad (54)$$

$$\textbf{Faraday} \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \quad (55)$$

$$\textbf{Gauss} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (56)$$

$$\textbf{Poisson} \quad \vec{\nabla} \cdot \vec{E} = 4\pi Q \quad (57)$$

(here we use strictly **Gauss units!!!**).

The **stress-energy tensor of the EM field** is defined as

$$\mathcal{T}^{\mu\nu}_{\rm EM} = -\frac{1}{4\pi}\,\mathcal{F}^{\mu\alpha}\mathcal{F}^{\nu}{}_{\alpha} + \frac{1}{16\pi}\,g^{\mu\nu}\,\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} \quad , \tag{58}$$

The following individual components of this stress-energy tensor can be identified with

- **the EM field energy density** $\mathcal{T}^{00}_{\rm EM} \equiv U_{\rm EM} = \frac{1}{8\pi}\left(E^2 + B^2\right)$
- **the EM field energy density (Poynting) flux** $c\mathcal{T}^{i0}_{\rm EM} \equiv P^{i}_{\rm EM} = \frac{c}{4\pi}\,(\vec{E}\times\vec{B})^i$
- **the EM field momentum flux density** $\mathcal{T}^{ik}_{\rm EM} \equiv \Pi^{ik}_{\rm EM} = -\frac{1}{4\pi}\left(E^i E^k + B^i B^k\right) + \frac{1}{8\pi}\left(E^2 + B^2\right)\,\delta^{ik}$

Note the two components in the momentum flux density (**"Maxwell stress"**) tensor, corresponding to the tension and pressure of the field lines, respectively. Also, $\mathcal{T}^{\mu\nu}_{\rm EM} = \mathcal{T}^{\nu\mu}_{\rm EM}$.

Recall the **Lorentz transformations** of four-vectors and tensors:

$$\mathcal{J}'^{\mu} = \Lambda^{\mu}{}_{\alpha}\, \mathcal{J}^{\alpha} \quad , \quad \mathcal{F}'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\, \mathcal{F}^{\alpha\beta} \tag{59}$$

where the Lorentz transformation matrix

$$\Lambda^{\mu}{}_{\alpha} = \begin{bmatrix} \Gamma & -\Gamma\beta_x & -\Gamma\beta_y & -\Gamma\beta_z \\ -\Gamma\beta_x & 1+\frac{\Gamma-1}{\beta^2}\beta_x^2 & \frac{\Gamma-1}{\beta^2}\beta_x\beta_y & \frac{\Gamma-1}{\beta^2}\beta_x\beta_z \\ -\Gamma\beta_y & \frac{\Gamma-1}{\beta^2}\beta_x\beta_y & 1+\frac{\Gamma-1}{\beta^2}\beta_y^2 & \frac{\Gamma-1}{\beta^2}\beta_y\beta_z \\ -\Gamma\beta_z & \frac{\Gamma-1}{\beta^2}\beta_x\beta_z & \frac{\Gamma-1}{\beta^2}\beta_y\beta_z & 1+\frac{\Gamma-1}{\beta^2}\beta_z^2 \end{bmatrix} \tag{60}$$

Keep in mind that

$$\mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{61}$$

It therefore follows that the relativistic transformation of the electric and magnetic field components are

$$\vec{E}' = \Gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) - \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{E}\right) \tag{62}$$

$$\vec{B}' = \Gamma\left(\vec{B} - \vec{\beta} \times \vec{E}\right) - \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{B}\right) \tag{63}$$

Note that **Lorentz transformations effectively "mix" the electric and magnetic field components**.

As for the electric charge density and currents, we have

$$cQ' = \Gamma\, cQ - \Gamma\left(\vec{\beta} \cdot \vec{j}\right) \tag{64}$$

$$\vec{j}' = \vec{j} - \Gamma\, cQ\,\vec{\beta} + \frac{\Gamma - 1}{\beta^2}\,\vec{\beta}\left(\vec{\beta} \cdot \vec{j}\right) \tag{65}$$

INTERLUDE

CLASSICAL OHM'S LAW

According to the **classical Ohm's law**, in the rest frame of a fluid the electric *conduction* current is

$$\vec{j}' = \sigma\,\vec{E}' \tag{66}$$

where $\sigma = e^2 n_e \tau_c / m_e$ is the fluid **conductivity**, for the electron number density n_e and the electron collision timescales with the bulk of fluid τ_c. Taking now the transformation of the EM field and currents in the non-relativistic regime, i.e., ignoring terms $\mathcal{O}(\beta^2)$ or higher, namely

$$\vec{E}' \simeq \vec{E} + \vec{\beta} \times \vec{B} \quad , \quad \vec{B}' \simeq \vec{B} - \vec{\beta} \times \vec{E} \quad , \quad \vec{j}' \simeq \vec{j} - \vec{v}Q \tag{67}$$

one therefore obtains

$$\frac{1}{\sigma}\left(\vec{j} - \vec{v}Q\right) \simeq \vec{E} + \vec{\beta} \times \vec{B} \tag{68}$$

meaning $\vec{E} \simeq -\vec{\beta} \times \vec{B}$ in the perfect conductivity limit, $\sigma^{-1} \to 0$.

The essential statement here is, in fact, that **the electric field must vanish in the fluid rest frame, if only the conductivity is infinite**,

$$\vec{E}' = 0 \quad \text{if} \quad \sigma^{-1} \to 0 \tag{69}$$

because in this limit charge carriers immediately rearrange to cancel all the rest-frame electric fields.

Assuming therefore the perfect conductivity limit, $\sigma^{-1} \to 0$, the **covariant form of the ideal Ohm's law** is

$$\mathcal{F}^{\mu\nu} u_\nu = 0 \tag{70}$$

This is not a full relativistic generalization of the Ohm's law, but only the covariant form assuring that, in the perfect conductivity limit, electric field is vanishing in the fluid rest frame. Indeed, note that in the fluid rest frame $\mathcal{F}'^{\mu\nu} u'_\nu = (0, \vec{E}')$, while in general $(u^\mu) = (\Gamma, \Gamma\,\beta^k)$, so that for the space components

$$\mathcal{F}^{i\nu} u_\nu = 0 \quad \to \quad \Gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right) = 0 \tag{71}$$

while the time component gives the consistency condition which is then automatically satisfied, namely

$$\mathcal{F}^{0\nu} u_\nu = 0 \quad \to \quad \Gamma\left(\vec{E} \cdot \vec{\beta}\right) = 0 \tag{72}$$

Note that for an ideal electric field $\vec{E} = -\vec{\beta} \times \vec{B}$, the Poynting flux becomes

$$\vec{P}_{\rm EM} = \frac{c}{4\pi}\left(\vec{E} \times \vec{B}\right) = \frac{c}{4\pi}\left[\left(\vec{\beta}\,B^2 - \vec{B}\,(\vec{\beta} \cdot \vec{B}\right)\right] \tag{73}$$

INTERLUDE

POYNTING THEOREM

Recall the identity which follows from the Maxwell's equations for given definitions of $\mathcal{T}^{\mu\nu}_{\mathrm{EM}}$ and $\mathcal{F}^{\mu\nu}$,

$$\nabla_\mu \mathcal{T}^{\mu\nu}_{\mathrm{EM}} = -\frac{1}{c}\,\mathcal{F}^{\nu\alpha}\,\mathcal{J}_\alpha \tag{74}$$

This relation describes the exchange of energy and momentum between the EM field and a matter, with the matter entering only through the 4-current $\mathcal{J}_\alpha$.

Let's consider first the time component of this identity, and in particular its both sides

$$\nabla_\mu \mathcal{T}^{\mu 0}_{\mathrm{EM}} = \frac{1}{c}\,\partial_t\, U_{\mathrm{EM}} + \vec{\nabla}\cdot\frac{1}{c}\vec{P}_{\mathrm{EM}} \quad \text{and} \quad -\frac{1}{c}\mathcal{F}^{0\alpha}\,\mathcal{J}_\alpha = -\frac{1}{c}\,\vec{j}\cdot\vec{E} \tag{75}$$

respectively. We have therefore

$$-\partial_t\, U_{\mathrm{EM}} = \vec{\nabla}\cdot\vec{P}_{\mathrm{EM}} + \vec{j}\cdot\vec{E} \tag{76}$$

i.e. the **Poynting theorem**, which can be also expressed in the integral form

$$-d_t \int U_{\mathrm{EM}}\, d\mathcal{V} = \int_{\partial\mathcal{V}} \vec{P}_{\mathrm{EM}}\cdot d\vec{\mathcal{S}} + \int \vec{j}\cdot\vec{E}\, d\mathcal{V} \tag{77}$$

The rate of changes of the EM field energy in a given volume is equal to the EM energy flowing in/out of the volume, minus the EM energy *dissipated* within this volume at the rate $\vec{j}\cdot\vec{E}$.

Now let's consider the space component of the $\nabla_\mu \mathcal{T}^{\mu\nu}_{\rm EM} = -\frac{1}{c}\,\mathcal{F}^{\nu\alpha}\,\mathcal{J}_\alpha$ identity, for which the both sides are

$$\nabla_\mu \mathcal{T}^{\mu k}_{\rm EM} = \frac{1}{c^2}\,\partial_t P^k_{\rm EM} + \nabla_i\,\Pi^{ki}_{\rm EM} \quad \text{and} \quad -\frac{1}{c}\mathcal{F}^{i\alpha}\,\mathcal{J}_\alpha = -Q\,\vec{E} - \frac{1}{c}\,\vec{j} \times \vec{B} \tag{78}$$

respectively. From this we obtain the equivalent of the momentum equation for the EM field, namely

$$\frac{1}{c^2}\,\partial_t \vec{P}_{\rm EM} + \vec{\nabla} \cdot \hat{\Pi}_{\rm EM} = -\vec{F}_{\rm L} \tag{79}$$

where the **Lorentz force** density is

$$\vec{F}_{\rm L} = Q\,\vec{E} + \frac{1}{c}\,\vec{j} \times \vec{B} \tag{80}$$

This is the **force exerted by the EM field on the fluid!**

The standard covariant formulation of relativistic ideal **magnetohydrodynamics (MHD)**, consists of

$$
\begin{aligned}
\text{total energy-momentum conservation} \qquad & \nabla_\mu \left(\mathcal{T}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{\mathrm{EM}} \right) = 0 && (81) \\
\text{particle number conservation} \qquad & \nabla_\mu \mathcal{D}^\mu = 0 && (82) \\
\text{Maxwell's equations (inhomogeneous)} \qquad & \nabla_\nu \mathcal{F}^{\mu\nu} = -\frac{4\pi}{c} \mathcal{J}^\mu && (83) \\
\text{Maxwell's equations (homogeneous)} \qquad & \epsilon^{\alpha\beta\mu\nu} \nabla_\beta \mathcal{F}_{\mu\nu} = 0 && (84) \\
\text{ideal Ohm's law} \qquad & \mathcal{F}^{\mu\nu} u_\nu = 0 && (85)
\end{aligned}
$$

Note that the charge conservation, $\nabla_\mu \mathcal{J}^\mu = 0$, is not an independent equation in this framework, but it follows automatically from Maxwell's equations, since $\nabla_\mu \nabla_\nu \mathcal{F}^{\mu\nu} \equiv 0$ for $\mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu}$.

MHD equations describe dynamics of a conducting fluid interacting with electromagnetic field.

Ideal MHD

Displacement current

Orders-of-magnitude analysis of the Faraday law implies

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\,\partial_t \vec{B} \quad \rightarrow \quad \frac{E}{\ell} \sim \frac{B\,v}{c\,\ell} \quad \rightarrow \quad E \sim \frac{v}{c}\,B \tag{86}$$

where ℓ is the characteristic spatial scale, and the dynamical timescale $\tau \sim \ell/v$ for the fluid velocity v. Hence, the **"displacement current"**

$$\frac{1}{c}\partial_t \vec{E} \quad \rightarrow \quad \frac{E\,v}{c\,\ell} \sim \left(\frac{v}{c}\right)^2 \frac{B}{\ell} \tag{87}$$

This implies in particular that, **in the non-relativistic regime** $v/c \ll 1$**, conductive currents** $\vec{j}$ **dominate the dynamics of the EM field**, since through the Amper's law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \quad \rightarrow \quad \frac{B}{\ell} \sim \frac{j}{c} \quad \gg \quad \left(\frac{v}{c}\right)^2 \frac{B}{\ell} \tag{88}$$

In the non-relativistic regime one can therefore neglect the $\mathcal{O}(\beta^2)$-order displacement current $c^{-1}\partial_t \vec{E}$.

Similarly, orders-of-magnitude analysis of the Poisson law implies

$$\vec{\nabla} \cdot \vec{E} = 4\pi Q \quad \rightarrow \quad \frac{E}{\ell} \sim Q \tag{89}$$

and therefore

$$Q\vec{E} \quad \rightarrow \quad \frac{E^2}{\ell} \sim \left(\frac{v}{c}\right)^2 \frac{B^2}{\ell} \tag{90}$$

This implies in particular that, **in the non-relativistic regime** $v/c \ll 1$**, currents also dominate the force exerted by the EM field on the fluid**, since

$$\frac{1}{c}\vec{j} \times \vec{B} \quad \rightarrow \quad \frac{j\,B}{c} \sim \frac{B^2}{\ell} \quad \gg \quad \left(\frac{v}{c}\right)^2 \frac{B^2}{\ell} \tag{91}$$

In the non-relativistic regime one can therefore neglect the $\mathcal{O}(\beta^2)$-order term $Q\vec{E}$ in the Lorentz force.

Let us therefore simplify Maxwell's equations by ignoring the displacement current, $c^{-1}\partial_t\vec{E}=0$, as well as by setting charge density to zero, $Q=0$, both of which assumptions are justified in the non-relativistic regime, as elaborated above. One then has in particular

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j}\quad,\quad\vec{\nabla}\times\vec{E}=-\frac{1}{c}\partial_t\vec{B}\quad,\quad\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot\vec{E}=0 \tag{92}$$

We also recall the non-relativistic Ohm's law $\sigma^{-1}\vec{j}=\vec{E}+\vec{\beta}\times\vec{B}$. By combining the above relations, and noting that $\vec{\nabla}\times(\vec{\nabla}\times\vec{B})=-\vec{\nabla}^2\vec{B}$, one obtains the equation governing an evolution of the magnetic field, aka the **magnetic induction equation**,

$$\partial_t\vec{B}=\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)+\eta\,\vec{\nabla}^2\vec{B} \tag{93}$$

where the **magnetic diffusivity** is

$$\eta\equiv\frac{c^2}{4\pi\sigma} \tag{94}$$

The first term on the right-hand side of this equation described **advection** of the magnetic field with the fluid, while the second term corresponds to the **diffusion** of the magnetic field out of the system.

Ideal MHD

Magnetic Reynolds number

Order-of-magnitude analysis of the two terms on the right-hand side of the magnetic induction equation:

$$\vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) \quad \rightarrow \quad \frac{B\, v}{\ell} \equiv \frac{B}{\tau_{\rm adv}} \quad \rightarrow \quad \tau_{\rm adv} = \frac{\ell}{v} \tag{95}$$

$$\eta\, \vec{\nabla}^2 \vec{B} \quad \rightarrow \quad \frac{B\, \eta}{\ell^2} \equiv \frac{B}{\tau_{\rm diff}} \quad \rightarrow \quad \tau_{\rm diff} = \frac{\ell^2}{\eta} \tag{96}$$

where ℓ is the characteristic spatial scale of the system. The **magnetic Reynolds number** is a ratio of the two corresponding timescales for the field advection and diffusion,

$$\mathcal{R}_{\rm M} \equiv \frac{\tau_{\rm diff}}{\tau_{\rm adv}} = \frac{\ell\, v}{\eta} \tag{97}$$

See that the regime of a high conductivity, $\sigma^{-1} \rightarrow 0$, implies $\eta \rightarrow 0$, meaning $\mathcal{R}_{\rm M} \rightarrow \infty$, or in other words $\tau_{\rm diff} \gg \tau_{\rm adv}$. That is, in the perfect conductivity limit, diffusion of the magnetic field out of the system, is negligible with respect to the advection of the magnetic field with the fluid.

For majority of cosmic plasmas, $\mathcal{R}_{\rm M} \gg 1$ **indeed!** In the case of the static system $v = 0$, the typical velocity that occurs in a magnetized fluid should be Alfvén velocity. With such, the magnetic Reynolds number is referred to as the **Lundquist number** .

Ideal MHD

Alfven's theorem

Assuming therefore perfect conductivity and non-relativistic bulk velocities, we have

$$\partial_t \vec{B} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) \tag{98}$$

which, keeping in mind the Gauss's law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$, implies the **magnetic flux conservation**, namely that the magnetic flux $\phi_{\mathrm{M}} = \int \vec{B} \cdot d\vec{\mathcal{S}}$ through a surface $\mathcal{S}$ *moving* with the bulk fluid velocity $\vec{v}$, is constant, $D_t\, \phi_{\mathrm{M}} = 0$ (**Alfven's theorem**, or the "frozen-in flux" theorem).

An another way of looking at it, is to use this simplified induction equation (with no diffusion term) combined with the continuity (mass conservation) equation $\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$, yielding

$$D_t\, \frac{\vec{B}}{\rho} = \left(\frac{\vec{B}}{\rho} \cdot \vec{\nabla} \right) \vec{v} \tag{99}$$

which implies the **field line conservation**, namely that changes of magnetic field per unit mass, B/ρ, along a fluid trajectory, are due only to the stretching or orientation of field lines caused by fluid motion.

Note first that that the magnetic flux through the volume $\mathcal{V}$ encompassed by a flow (fluid element) moving with velocity $\vec{v}$ in a time interval $(t, t+dt)$, is

$$\phi_M(t+dt)|_{S'} - \phi_M(t+dt)|_S + \phi_M(t+dt)|_A$$
$$= \int_{\partial\mathcal{V}} \vec{B}(t+dt)\, d\vec{\mathcal{S}} = \int \vec{\nabla}\cdot\vec{B}(t+dt)\, d\mathcal{V} = 0 \quad (100)$$

where in the second line we used the Gauss theorem (volume/surface integral) and next the Gauss law ($\vec{\nabla}\cdot\vec{B} = 0$). Note also that

$$\phi_M(t+dt)|_A \equiv \int \vec{B}(t)\cdot d\vec{A} \quad (101)$$
$$= \oint_{\partial S} \vec{B}(t)\cdot\left(d\vec{l}\times\vec{v}\,dt\right) \quad (102)$$

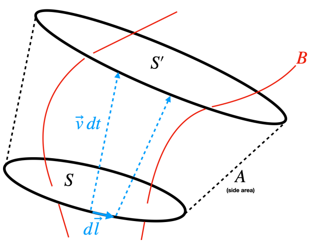
On the other hand, we have by definition

$$D_t\,\phi_M = \phi_M(t+dt)|_{S'} - \phi_M(t)|_S \quad (103)$$

Hence

$$\begin{aligned} D_t\,\phi_M &= \phi_M(t+dt)|_S - \phi_M(t)|_S \quad (104)\\ &\quad - \oint_{\partial S} \vec{B}(t)\cdot\left(d\vec{l}\times\vec{v}\,dt\right)\\ &\equiv \partial_t\phi_M - \oint_{\partial S}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{l}\\ &= \int\left[\partial_t\vec{B} - \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)\right]\cdot d\vec{\mathcal{S}} = 0 \end{aligned}$$

where we used the vector identity $\vec{B}\cdot(\vec{A}\times\vec{C}) = (\vec{C}\times\vec{B})\cdot\vec{A}$, next the Stoke's theorem $\oint_{\partial S}\vec{X}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{X})\cdot d\vec{\mathcal{S}}$, and finally the simplified magnetic induction equation.



Ideal MHD

Total energy conservation

Let us now consider the time component of the total energy-momentum conservation, i.e., the energy equation $\nabla_\mu \left(\mathcal{T}^{\mu 0} + \mathcal{T}^{\mu 0}_{\mathrm{EM}}\right) = 0$. In Cartesian coordinates it can be written in the **conservation form** as

$$\partial_t \left(U + U_{\mathrm{EM}}\right) + \vec{\nabla} \cdot \left[(U + p)\vec{v} + \vec{P}_{\mathrm{EM}}\right] = 0 \tag{105}$$

where, as defined previously, the fluid total energy density $U = w\Gamma^2 - p$, the EM field energy density $U_{\mathrm{EM}} = (E^2 + B^2)/8\pi$, and the Poynting flux $\vec{P}_{\mathrm{EM}} = c\,(\vec{E} \times \vec{B})/4\pi$.

Using the Poynting theorem $\partial_t U_{\mathrm{EM}} + \vec{\nabla} \cdot \vec{P}_{\mathrm{EM}} = -\vec{j} \cdot \vec{E}$ to eliminate the EM energy terms, we find the fluid energy equation becomes

$$\partial_t U + \vec{\nabla} \cdot (U + p)\vec{v} = \vec{j} \cdot \vec{E} \tag{106}$$

However, **for an ideal fluid, there can be no energy dissipation, including any conversion of EM energy into the internal energy of the fluid!** In other words, in ideal MHD, since $\vec{j} \cdot \vec{E} = 0$, the energy equation reduces to a conservation equation for the fluid's total (bulk + internal) energy.

Ideal MHD

Ohmic dissipation

Recall that in an ideal (non-dissipative), non-relativistic fluid, conservation of internal energy, $\partial_t \tilde{U} + \vec{\nabla} \cdot (\tilde{U} + p)\vec{v} = 0$, leads to the conservation of specific entropy, $D_t s = 0$. **Since entropy increases only through irreversible processes, a non-zero $\vec{j} \cdot \vec{E}$ in the analyzed conservation of internal energy equation, would imply $D_t s \neq 0$.**

Now, remember that the generalized Ohm's law is $\vec{E} \simeq \sigma^{-1} \vec{j} - \vec{\beta} \times \vec{B}$, and hence

$$\vec{j} \cdot \vec{E} = \frac{1}{\sigma} j^2 - \vec{j} \cdot (\vec{\beta} \times \vec{B}) = \frac{1}{\sigma} j^2 + \vec{v} \cdot \vec{F}_{\mathrm{L}} \tag{107}$$

where $\vec{F}_{\mathrm{L}} = c^{-1} \vec{j} \times \vec{B}$ is the Lorentz force. That is, in the ideal MHD limit $\sigma^{-1} \to 0$ and $\vec{j} \cdot \vec{E} = 0$, the work done by the electromagnetic field is entirely reversible mechanical work, $\vec{v} \cdot \vec{F}_{\mathrm{L}} = 0$, manifesting as plasma bulk acceleration, with no irreversible heating, and as such is fully accounted for by the momentum conservation equation: fluid and field exchange momentum, but not energy.

Only when resistivity is finite, $\sigma^{-1} \neq 0$, can Ohmic dissipation (Joule's heating) occur; plasma resistivity provides then a dissipative sink for magnetic field energy: **as magnetic energy diffuses out of the system, it decreases over time, and the entire loss is converted into Ohmic heating of the fluid, with $\rho T \, D_t s = \sigma^{-1} j^2$.**

IDEAL MHD

TOTAL MOMENTUM CONSERVATION

In an analogous way, when considering space components of the total energy-momentum conservation, i.e., the momentum equation $\nabla_\mu \left(\mathcal{T}^{\mu k} + \mathcal{T}^{\mu k}_{\mathrm{EM}}\right) = 0$, we obtain the **conservation form**

$$\partial_t \left(P^k + \frac{1}{c^2} P^k_{\mathrm{EM}} \right) + \partial_i \left(P^k v^i + p\delta^{ik} + \Pi^{ki}_{\mathrm{EM}} \right) = 0 \tag{108}$$

where the Maxwell stress tensor $\Pi^{ik}_{\mathrm{EM}} = -\frac{1}{4\pi}\left(E^i E^k + B^i B^k\right) + \frac{1}{8\pi}\left(E^2 + B^2\right)\delta^{ik}$. Using the momentum equation for EM field, the above is equivalent to

$$\partial_t P^k + \partial_i \left(P^k v^i + p\delta^{ik} \right) = F^k_{\mathrm{L}} \tag{109}$$

From here it follows that the EM field is acting dynamically on the fluid through the Lorentz force $\vec{F}_{\mathrm{L}}$. **This leads to changes in the fluid's bulk kinetic energy and/or pressure, but does not increase its entropy.**

Neglecting the displacement current ($c^{-1}\partial_t\vec{E} = 0$) and assuming electric neutrality of a fluid ($Q = 0$) — both of which are justified in the non-relativistic regime, as elaborated above — the Lorentz force becomes

$$\begin{aligned}\vec{F}_{\mathrm{L}} = \frac{1}{c}\vec{j}\times\vec{B} &= \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = \\ &= \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right) .\end{aligned} \tag{110}$$

The first term on the right-hand side represents the magnetic tension force, which acts along curved field lines ("tension pulls"), while the second term corresponds to the magnetic pressure gradient, which acts perpendicular to field lines where magnetic pressure varies ("pressure pushes").

Ideal MHD

Toroidal Magnetic Field

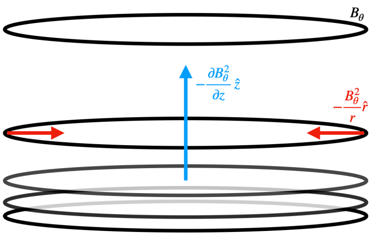
As an example to help us better understand the magnetic tension term $(\vec{B}\cdot\vec{\nabla})\vec{B}$ and the magnetic pressure term $-\vec{\nabla}B^2$ in the Lorentz force, let us consider an axisymmetric magnetic field in cylindrical coordinates (r,θ,z) **consisting of a purely toroidal component with a possible gradient in the z-direction**. Specifically, we take

$$\vec{B} = B_r\,\hat{r} + B_\theta\,\hat{\theta} + B_z\,\hat{z}, \quad \text{with}$$
$$B_\theta = B_\theta(z),\ B_r = B_z = 0 \tag{111}$$

i.e., a field with only a toroidal component $B_\theta\hat{\theta}$, which may vary along $\hat{z}$, and no poloidal component ($B_r = B_z = 0$). For such a field, we have:

$$(\vec{B}\cdot\vec{\nabla})\vec{B} = -\frac{B_\theta^2}{r}\hat{r} \tag{112}$$

$$-\vec{\nabla}B^2 = -\frac{\partial B_\theta^2}{\partial z}\,\hat{z} \tag{113}$$



The first term thus corresponds to a radially inward force ($-\hat{r}$) due to the curvature of the magnetic field lines (magnetic tension), while the second represents a vertical force in the $\hat{z}$-direction resulting from a gradient in the magnetic energy density/magnetic pressure.

Ideal MHD

Ideal non-relativistic MHD: main assumptions

Let's clarify the main assumptions behind "non-relativistic ideal MHD approximation":

- **non-relativistic bulk velocities** $\beta \ll 1$

  $\rightarrow \quad \partial_t \vec{E} \ll 4\pi \vec{j} \quad$ and $\quad Q\vec{E} \ll c^{-1}\, \vec{j} \times \vec{B}$

  $\rightarrow \quad \vec{j} \simeq (c/4\pi)\,(\vec{\nabla} \times \vec{B}) \quad$ and $\quad \vec{F}_{\mathrm{L}} \simeq c^{-1}\, \vec{j} \times \vec{B}$

- **perfect conductivity regime** $\sigma^{-1} \rightarrow 0$

  $\rightarrow \quad \vec{E}' = 0$

  $\rightarrow \quad \vec{E} \simeq -\vec{\beta} \times \vec{B} \quad$ and $\quad \vec{j} \cdot \vec{E} = 0$

- **both** $\beta \ll 1$ **and** $\sigma^{-1} \rightarrow 0$

  $\rightarrow \quad \mathcal{R}_{\mathrm{M}} \gg 1$

  $\rightarrow \quad \partial_t \vec{B} \simeq \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$

All in all, the set of **non-relativistic ideal MHD equations for a polytropic fluid** reads:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{114}$$

$$D_t \left(p / \rho^{\hat{\gamma}} \right) = 0 \tag{115}$$

$$\rho \, D_t \vec{v} = -\vec{\nabla} p + \frac{1}{c} \vec{j} \times \vec{B} \tag{116}$$

$$\partial_t \vec{B} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) \tag{117}$$

where $\vec{j} \simeq c\,(\vec{\nabla} \times \vec{B})/4\pi$, and the solenoidal constraint $\vec{\nabla} \cdot \vec{B} = 0$ is imposed as an initial and boundary condition. Remember that conservation of energy along the flow, eq. 115, is equivalent to the conservation of specific entropy, $T\, D_t\, s = 0$, for a polytropic fluid with the equation of state $p/\rho^{\hat{\gamma}} = K(s)$.

In the ideal non-relativistic MHD, electric field becomes a secondary quantity!

Let us define the **plasma magnetization parameters** $\sigma_{\rm pl}$ **and** $\beta_{\rm pl}$ as ratios of fluid bulk kinetic energy density $\frac{1}{2}\rho v^2$, or pressure p, to the magnetic field energy density $U_B = B^2/8\pi$,

$$\sigma_{\rm pl} \equiv \frac{U_B}{\frac{1}{2}\rho v^2} \quad \text{and} \quad \beta_{\rm pl} \equiv \frac{p}{U_B}\,. \tag{118}$$

Note that, up to orders-of-magnitude, relative magnitudes in various terms in the ideal MHD (non-relativistic) momentum equation $\rho\, D_t\vec{v} = -\vec{\nabla}p + c^{-1}\,\vec{j}\times\vec{B}$, are

$$\frac{\left|c^{-1}\,\vec{j}\times\vec{B}\right|}{\left|\rho\, D_t\vec{v}\right|} \to \sigma_{\rm pl} \quad \text{and} \quad \frac{\left|\vec{\nabla}p\right|}{\left|c^{-1}\,\vec{j}\times\vec{B}\right|} \to \beta_{\rm pl} \tag{119}$$

Therefore, the condition $\sigma_{\rm pl} \gg 1$ corresponds to a regime where magnetic forces dominate inertia, and the momentum equation approaches a **magnetohydrostatic force balance:** $\vec{\nabla}p \simeq (1/c)\,\vec{j}\times\vec{B}$.
If, in addition, $\beta_{\rm pl} \ll 1$, then the pressure gradient becomes negligible, and we approach the **force-free magnetic field limit:** $\vec{j}\times\vec{B} \simeq 0$. In this regime, the magnetic field is in equilibrium with itself, with a balance between magnetic pressure and tension, and currents flow along field lines: $\vec{j}\|\vec{B}$.

Ideal MHD

Ideal non-relativistic MHD: conservation form

The system of ideal non-relativistic MHD equations, can be cast in the **conservation form**, where we ignore the electric terms in the Maxwell stress tensor $\hat{\Pi}_{\rm EM}$ and in the field energy density $U_{\rm EM}$, as well as the time variation of the Poynting flux $\partial_t \vec{P}_{\rm EM}$, as these are all of the $\mathcal{O}(v/c)^2$ order. As a result, we obtain:

$$\partial_t \rho \quad + \quad \partial_i \left[\rho v^i\right] = 0 \tag{120}$$

$$\partial_t \left(\frac{1}{2}\rho v^2 + \varepsilon + \frac{1}{8\pi}B^2\right) \quad + \quad \partial_i \left[\left(\frac{1}{2}\rho v^2 + \varepsilon + p + \frac{1}{8\pi}B^2\right) v^i - \frac{1}{4\pi} v_j B^i B^j\right] = 0 \tag{121}$$

$$\partial_t \left(\rho v^k\right) \quad + \quad \partial_i \left[\rho v^i v^k + \left(p + \frac{1}{8\pi}B^2\right)\delta^{ik} - \frac{1}{4\pi}B^i B^k\right] = 0 \tag{122}$$

$$\partial_t B^k \quad + \quad \partial_i \left[v^i B^k - v^k B^i\right] = 0 \tag{123}$$

Keeping in mind equation of state that relates w directly with n and p, as well as ideal Ohm's law relating $\vec{E}$ with $\vec{\beta}$ and $\vec{B}$, we have 8 dynamical variables: n, p, three components of $\vec{\beta}$, and three components of $\vec{B}$; for these, we have eight ideal MHD equations to consider: particle flux conservation, energy conservation, three equations for the momentum conservation, and three from the magnetic induction (Faraday's law). However we have also one "boundary condition" equation provided by the Gauss law for magnetism, meaning that we do have eight equations, but with 7 degrees of freedom.

Part III

Waves and Shocks

MHD WAVES

SOUND AND ALFVEN SPEEDS

Let us define first the **sound speed** as

$$c_s = c\sqrt{\frac{\hat{\gamma}\,p}{w}} \xrightarrow{\text{non-relat. fluid: } w \simeq \rho c^2} \sqrt{\frac{\hat{\gamma}\,p}{\rho}} \tag{124}$$

and the **Alfven speed** as

$$v_A = c\sqrt{\frac{B^2/4\pi}{w + B^2/4\pi}} \xrightarrow[w \gg U_B]{\text{non-relat. fluid: } w \simeq \rho c^2} \frac{B}{\sqrt{4\pi\rho}} \tag{125}$$

Note that the plasma magnetization parameters are then $\sigma_{\rm pl} = (v/v_A)^2$ and $\beta_{\rm pl} \simeq (c_s/v_A)^2$.

In hydrodynamics, if the sound speed c_s is much faster than the fluid bulk velocity v (i.e. if the flow is extremely subsonic), changes in the fluid density can be approximately treated as being instantaneous, and so the flow can be considered as incompressible: $\vec{\nabla} \cdot \vec{v} = 0$.

MHD WAVES

INTRODUCING PERTURBATIONS

We perturb the system of ideal non-relativistic MHD equations (114)–(117) by introducing **small variations of the unperturbed quantities**:

$$\vec{B} = \vec{B}_0 + \vec{B}_1 , \quad \vec{v} = \vec{v}_0 + \vec{v}_1 , \quad \rho = \rho_0 + \rho_1 , \quad p = p_0 + p_1 . \tag{126}$$

We then **linearize the equations**, assuming that the perturbations are small: $\rho_1/\rho_0 \ll 1$, etc., and consider a single Fourier component of each perturbation:

$$\rho_1 \propto \exp[i(\vec{k} \cdot \vec{x} - \omega t)], \quad \text{etc.} \tag{127}$$

That is, **we assume the perturbations are plane waves** propagating with frequency $\omega/(2\pi)$ in the direction of the wavevector $\vec{k}$.
The linearized, Fourier-transformed system forms a closed, linear, and homogeneous set of equations for the unknowns: $\vec{B}_1$, $\vec{v}_1$, ρ_1, and p_1, with $\vec{k}$ and ω treated as parameters (determined by the initial or boundary conditions). To obtain non-trivial solutions, we set the determinant of the coefficient matrix to zero; this yields the **dispersion relation** for the perturbation waves:

$$\omega = \omega(\vec{k}) . \tag{128}$$

MHD WAVES

DISPLACEMENT VECTOR

Alternatively, we may introduce the **displacement vector** $\vec{\xi}$, which gives the direction and distance a fluid element is displaced from its equilibrium position:

$$\vec{\xi}(\vec{r}, t=0) = 0 \quad \text{and} \quad \vec{\xi}(\vec{r}, t) = \int_0^t \vec{v}_1(\vec{r}, t')\, dt' \tag{129}$$

so that

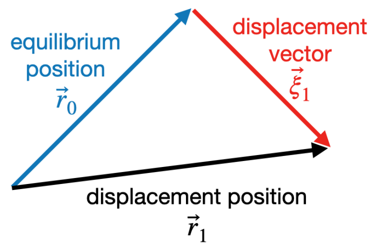
$$\partial_t \vec{\xi} = \vec{v}_1(\vec{r}, t) \tag{130}$$

From the linearized ideal MHD equations, by replacing $\vec{v}_1$ with $\partial_t \vec{\xi}$, we can then derive:

$$\rho_0\, \partial_t^2 \vec{\xi} = \vec{\mathcal{F}} \tag{131}$$

where $\vec{\mathcal{F}} = \vec{\mathcal{F}}(\vec{\xi}; p_0, \vec{B}_0)$ is the force acting on a fluid element when it is displaced by $\vec{\xi}$. Again, we look for **wave-like solutions** $\vec{\xi} \propto \exp[i(\vec{k} \cdot \vec{x} - \omega t)]$, obtaining:

$$\begin{aligned} \rho_0 \omega^2 \vec{\xi} \;=\; & \vec{k}\, \hat{\gamma} p_0\, (\vec{k} \cdot \vec{\xi}) + \\ & \frac{1}{4\pi} \left\{ \vec{k} \times \left[\vec{k} \times \left(\vec{\xi} \times \vec{B}_0 \right) \right] \right\} \times \vec{B}_0 \end{aligned} \tag{132}$$



One may note that the condition $\vec{\xi} \cdot \vec{\mathcal{F}} \lesseqgtr 0$ determines whether instabilities caused by the perturbation grow; a negative sign indicates a restoring force, while a positive sign implies amplification and instability.

MHD WAVES

DISPERSION RELATION

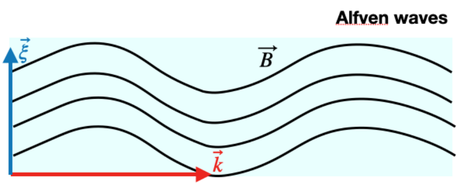
All in all, after **transforming to the frame moving with plasma**, $\omega_0 = \omega - \vec{k} \cdot \vec{v}_0$, we obtain

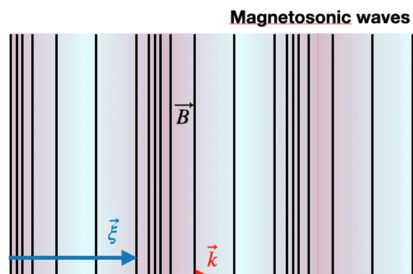
$$\omega_0^2 \left[\omega_0^2 - \left(\vec{k} \cdot \vec{v}_A \right)^2 \right] \left[\omega_0^4 - k^2 \left(c_s^2 + v_A^2 \right) \omega_0^2 + k^2 c_s^2 \left(\vec{k} \cdot \vec{v}_A \right)^2 \right] = 0 \,, \tag{133}$$

with $\vec{v}_A \equiv v_A \hat{B}_0$, so that $\vec{k}\, \vec{v}_A = k\, v_A \cos\phi$, where ϕ is the angle between the unperturbed magnetic field $\vec{B}_0$ and the direction of the wave propagation $\vec{k}$.

From there one may calculate the **phase and group velocities, i.e., velocities of single plane waves and wave packets, respectively**:

$$v_{ph} = \frac{\omega}{k} \quad \text{and} \quad v_{gr} = \frac{\partial \omega}{\partial k} \,. \tag{134}$$





MHD WAVES

ALFVEN WAVES

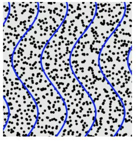
The solution

$$\omega_0^2 = k^2 v_A^2 \cos^2\phi \tag{135}$$

describes **Alfven waves** (AWs), propagating in the '$+$' or '$-$' direction at some angle ϕ to the unperturbed magnetic field B_0 with the phase velocity $v_A \cos\phi$ (as measured in the fluid rest-frame). This is a displacement of the plasma element together with the magnetic field frozen into it.

Properties:

- *incompressive*: there are no density, pressure, or entropy fluctuations, $\rho_1 = p_1 = s_1 = 0$;
- *transversal*: the fluid velocity perturbations and the magnetic field perturbations are transverse to both the direction of propagation and the unperturbed magnetic field, $\vec{v}_1 \perp \vec{k}, \vec{B}_0$, and $\vec{B}_1 \perp \vec{k}, \vec{B}_0$;
- *nondispersive*: all frequencies propagate at the same speed.



Movie: Standing Alfven wave (Spruit, 2013)

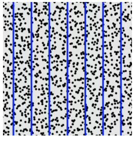
The solution

$$\omega_0^4 - \omega_0^2\, k^2 \left(c_s^2 + v_A^2\right) + k^4\, c_s^2\, v_A^2\, \cos^2\phi = 0\,. \tag{136}$$

gives **slow magnetosonic waves and fast magnetosonic waves** (SMWs and FMWs). The phase velocities of these modes are

$$v^2_{\mathrm{ph,\,f/s}} = \frac{1}{2}\left(c_s^2 + v_A^2\right) \pm \frac{1}{2}\sqrt{\left(c_s^2 + v_A^2\right)^2 - 4\, c_s^2\, v_A^2\, \cos^2\phi}\,. \tag{137}$$

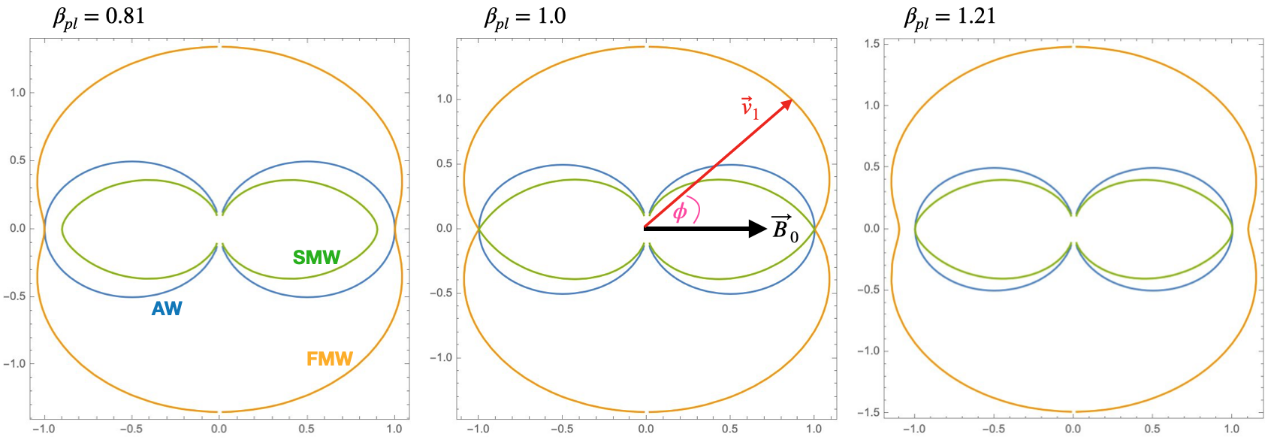
These modes involve compressions and rarefactions of both the plasma and the frozen-in magnetic field. The perturbed fluid velocity lies in the plane defined by $\vec{k}$ and $\vec{B}_0$. In the fast mode, magnetic and thermal pressure perturbations reinforce one another, while in the slow mode they act in opposition.



Movie: Standing slow mode for $v_A/c_s = 0.5$ and $\phi = 0.4$ rad (Spruit, 2013)

MHD waves

Phase velocity Friedrichs diagrams



MHD WAVES

LIMITS

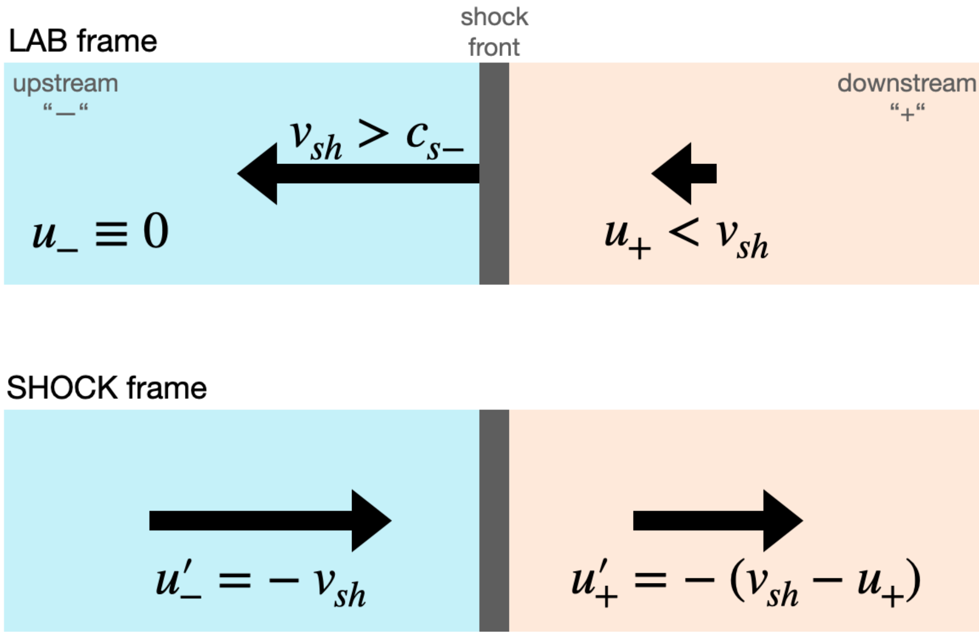
- In the “zero temperature” limit, i.e. for negligible fluid pressure, FMWs become **compressible Alfven waves** (CAWs), which propagate in all directions (i.e., for all angles ϕ) with velocity perturbations $\vec{v}_1 \cdot \vec{k} \neq 0$, non-vanishing density perturbations $\rho_1 \neq 0$, and with the dispersion relation $\omega_0^2 = k^2 v_{\mathrm{A}}^2$.
- In the case of vanishing magnetic field, magnetosonic modes become ordinary sound waves (SWs); these longitudinal waves ($\vec{v}_1 \parallel \vec{k}$) satisfy the wave equation $\partial_t^2 p_1 = c_s^2 \nabla^2 p_1$ giving the dispersion relation $\omega_0^2 = c_s^2 k^2$, and as such propagate equally in all directions with phase and group velocity c_s.
- In other words, with negligible pressure but non-negligible magnetic field, we have AWs and CAWs; on the other hand, with non-negligible pressure but negligible magnetic field, we have only SWs. When both gaseous pressure and magnetic field are considered (ideal MHD), AWs remain unaffected, but CAWs and SWs couple to form SMWs and FMWs.
- In the regime of a strong magnetic field $w \ll U_B$, we have $v_{\mathrm{A}} \simeq c$. Meanwhile, for a fluid with relativistic equation of state $w = 4p$, one has $c_s = c/\sqrt{3}$.
- The solution $\omega_0^2 = 0$ in equation 133, or equivalently $\omega - \vec{k}\,\vec{v}_0 = 0$, describes **entropy waves** (EWs); these correspond to a degenerate (zero-frequency) mode in the MHD wave spectrum, with phase velocity $v_{\mathrm{ph}} = \omega_0/k = 0$ in the fluid rest frame, typically caused by initial spatial variations in entropy or pressure/density *without* corresponding variations in velocity or magnetic field (a moving plasma carries the entropy waves with it – passive advection of thermodynamic inhomogeneities).

In collisional fluid dynamics, **shock waves** are a type of **contact discontinuity** where the flow macroscopic parameters — bulk velocity, pressure, density — change abruptly due to supersonic motion. In general, such surfaces may propagate with velocities different from the fluid bulk velocity. Unlike **tangential or rotational discontinuities**, which involve no flow across the surface and conserve energy, **shocks allow for the flow of matter across the discontinuity and are inherently dissipative, converting kinetic energy into internal energy and increasing entropy**. They represent a breakdown of smooth flow and are fully allowed in the framework of compressible hydrodynamics.

In collisionless magnetized plasmas, shocks can also form when the flow speed exceeds the fast magnetosonic speed. In contrast to collisional shocks, where dissipation arises from molecular collisions, dissipation in collisionless shocks occurs over spatial scales comparable to the ion or electron kinetic scales (such as the gyroradius or inertial length) and is mediated by collective plasma processes, including wave-particle interactions and electromagnetic instabilities.

Shocks

Shock rest frame



Shocks

Relativistic hydro shocks

Consider a flow along the $\hat{x}$ direction, and a shock front perpendicular to $\vec{\beta}$. The relevant conservation laws are $\nabla_\mu \mathcal{T}^{\mu\nu} = 0$ and $\nabla_\mu \mathcal{D}^\mu = 0$. Assuming a steady state ($\partial_t \to 0$), these reduce to

$$\nabla_1 \mathcal{T}^{10} = 0, \quad \nabla_1 \mathcal{T}^{1k} = 0, \quad \text{and} \quad \nabla_1 \mathcal{D}^1 = 0 \tag{138}$$

Integrating these equations over a cylindrical volume aligned along $\hat{x}$ and crossing the shock front, and applying Gauss's theorem, we note that the surface integrals over the lateral sides of the cylinder vanish due to $d\vec{\mathcal{S}} \cdot \vec{\beta} = 0$. This yields the following **jump conditions across the shock**:

$$\mathcal{T}^{10}_- = \mathcal{T}^{10}_+ \quad \rightarrow \quad w_- \Gamma^2_- \beta_- = w_+ \Gamma^2_+ \beta_+ \tag{139}$$

$$\mathcal{T}^{1k}_- = \mathcal{T}^{1k}_+ \quad \rightarrow \quad w_- \Gamma^2_- \beta^2_- + p_- = w_+ \Gamma^2_+ \beta^2_+ + p_+ \tag{140}$$

$$\mathcal{D}^1_- = \mathcal{D}^1_+ \quad \rightarrow \quad n_- \Gamma_- \beta_- = n_+ \Gamma_+ \beta_+ \tag{141}$$

These equations express the **conservation of energy flux, momentum flux, and particle flux across the shock front**, respectively.

Shocks

Non-relativistic hydro shocks

In the non-relativistic limit with no magnetic field, the shock jump conditions read as

$$\rho_-\, v_-^2 + p_- = \rho_+\, v_+^2 + p_+ \tag{142}$$

$$\frac{\varepsilon_- + p_-}{\rho_-} + \frac{1}{2}\, v_-^2 = \frac{\varepsilon_+ + p_+}{\rho_+} + \frac{1}{2}\, v_+^2 \tag{143}$$

$$\rho_-\, v_- = \rho_+\, v_+ \tag{144}$$

(**Rankine-Hugoniot conditions**). Thus, for the adiabatic index $\hat{\gamma}$ the same for the preshock (upstream) and the postshock (downstream) regions, one obtains

$$R \equiv \frac{\rho_+}{\rho_-} = \frac{v_-}{v_+} = \frac{(\hat{\gamma}+1)\,\mathcal{M}_-^2}{(\hat{\gamma}-1)\,\mathcal{M}_-^2 + 2} \tag{145}$$

$$\frac{p_+}{p_-} = \frac{2\,\hat{\gamma}\,\mathcal{M}_-^2}{\hat{\gamma}+1} - \frac{\hat{\gamma}-1}{\hat{\gamma}+1} \tag{146}$$

where the **upstream Mach number** is

$$\mathcal{M}_- \equiv \frac{v_-}{c_{\mathrm{s},\,-}} \tag{147}$$

Hence, in the regime of a strong shock $\mathcal{M}_- \gg 1$,

$$R \xrightarrow{\mathcal{M}_- \gg 1} \frac{\hat{\gamma}+1}{\hat{\gamma}-1} \xrightarrow{\hat{\gamma}=5/3} 4 \tag{148}$$

Shocks

Parallel MHD shocks

For the case of a parallel shock — i.e. when $\vec{B}$ lies entirely along the $\hat{x}$ direction and is therefore parallel to the shock normal $\hat{n}$ — we apply Gauss's law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$ to a cylindrical volume aligned along $\hat{x}$ and crossing the shock front. Integrating over this volume and using Gauss's theorem, we obtain:

$$\int d\mathcal{V} \left(\vec{\nabla} \cdot \vec{B}\right) = \int_{\partial\mathcal{V}} \vec{B} \cdot d\vec{\mathcal{S}} = -\int_{\mathcal{S}_0} B_- \, d\mathcal{S} + \int_{\mathcal{S}_0} B_+ \, d\mathcal{S} = 0 \quad \rightarrow \quad B_- = B_+ \tag{149}$$

That is, **the component of the magnetic field parallel to the shock normal remains continuous across the shock**. More generally, this condition can be written as:

$$[\hat{n} \cdot \vec{B}]_- = [\hat{n} \cdot \vec{B}]_+ \tag{150}$$

parallel shock

Shocks

Perpendicular MHD shocks

For the case of a perpendicular shock — i.e. when $\vec{B}$ lies entirely in the $\hat{y} - \hat{z}$ plane and is therefore perpendicular to the shock normal $\hat{n}$ — we consider the steady-state induction equation $\vec{\nabla} \times \vec{E} = 0$ keeping in mind that $\vec{E} = -(\vec{v}/c) \times \vec{B}$; integrating over a rectangular area spanning the shock and aligned along $\hat{x}$, and applying Stokes' theorem, we find that the tangential electric field must be continuous across the shock:

$$\int d\mathcal{S} \left(\vec{\nabla} \times \vec{E}\right) = \int_{\partial\mathcal{S}} \vec{E} \cdot d\ell = +\int_{\ell_0} E_- \, d\ell - \int_{\ell_0} E_+ \, d\ell = 0 \quad \rightarrow \quad E_- = E_+ \tag{151}$$

the continuity of $\vec{E}$ implies that **the transverse magnetic field is compressed in proportion to the velocity jump**: $B_+ = (v_-/v_+)\, B_- \equiv R\, B_-$; More generally, this continuity condition can be written as:

$$[\hat{n} \times (\vec{v} \times \vec{B})]_- = [\hat{n} \times (\vec{v} \times \vec{B})]_+ \tag{152}$$

perpendicular shock